**General Relativity Geometry**

**Einstein’s Field Equations**

Should refer to that Tensors folder for context…Following up on the insight suggested by the first postulate, we want to determine the curvature of space-time, and ultimately, the metric. So we’ll follow Einstein and make a heuristic inference as to what exactly determines the curvature. First, the curvature of space-time is described by the Riemann curvature tensor, Rαβδγ, and by its various contractions: the Ricci tensor Rαβ = Rδαβδ and the Ricci scalar R = Rαα. Seeking the simplest possible equations of gravity, we would first consider the possibility of having a scalar equation of gravity:



Now Newton’s theory of gravity states that



and so we might suspect that it is the mass density that determines the curvature. But knowing that mass is convertible into energy and vice versa we should probably say rather that it is the rest energy density that determines the curvature. So we might postulate:



where ρ0 is the mass density causing the gravitational field. But such an equation is a little wanting – for instance it places the rest frame of the mass as the preferred frame and we should have an equation which doesn’t give preference to any coordinate system. And in any case, such an equation was tested initially but was found to make false predictions – namely it didn’t correctly predict the advance of the perihelion of Mercury; instead it predicted it should retrograde. So going to the next higher up equation we could have:



The right hand side must be a tensor, and a natural candidate which includes the object’s mass, energy, etc., is the stress-energy tensor Tαβ. So we might postulate:



where,



But this can’t be quite right because we know that the stress-energy tensor satisfies the equation:



while Rαβ does not, generally. But this isn’t the end – we can perhaps add to the LHS other terms which would make the space-time divergence equal to 0. So we’ll add to the LHS other terms which are related to the curvature:



Now let’s see if we can choose the coefficients to make the LHS satisfy the divergence identity. We want:



First let’s show that the 4-divergence of is zero regardless.



And so now we have:



And now let’s work on these last terms. To help us out, we’ll use the Bianchi identity:



Applying the metric to both sides, and using the fact that ’s derivative is zero we can write:



This is called the first contracted Bianchi identity. Now let’s contract again over the β and ν indices.



This is called the twice-contracted Bianchi identity by the way. So filling this into our equation above we get:



So now that we have the required μ, we have an acceptable set of equations called the Einstein equation:



The constant Λ is called the cosmological constant, which Einstein originally discarded for aesthetic reasons, put back in to produce a static universe, and then took out again after it was discovered the universe is actually expanding and not static. It has been (well, not so) recently put back in *again* because it has been found the universe’s rate of expansion is actually *accelerating*. Λ is considered to be the source term for dark energy, which has/had been thought to be due to quantum field vacuum fluctuations. In that spirit we’d write this as:



But estimates of Λ based on this idea don’t match up at all (they are 100 orders of magnitude too large) with experimental measurements.

The constant k would be so far undetermined. But it can be inferred by calculating the field equations for weak gravitational fields (the Newtonian limit) and choosing k to match Newton’s gravitational field equations. We’ll see how this is done in a bit. Another point, the stress-energy tensor has been previously discussed for matter, and is given by (recall ρ0 is rest density, and ε0 = ρ0c2 is rest energy density):



for dust, and



for ‘fluids’ [note we updated → ]. But you will know upon a study of EM, fields also have energy and momentum, and therefore a stress-energy tensor of their own, which is defined in the same way. And so we should include this in **T** as well. So for instance, the stress-energy tensor for EM fields is (see EM folder/Symmetries and Conservation Laws):



For future reference, we can put the Ricci scalar in terms of the trace of the stress-energy tensor as follows.



So we pause to note this interesting fact, that the curvature of space-time is governed by the trace of the energy tensor, and the cosmological constant.



and this allows us to write Einstein’s equation as:



so,



Should also mention that there is a more elegant formulation of the field equations due to David Hilbert (I think). This is a Lagrangian approach based on minimizing an action (of course). S is given by:



R is the curvature. Interesting that R is on par with the energy in the fluid flow guy. So R is commensurate with energy somehow. And we’ll remember that √(-det g) is the prefactor that gives the volume element when multiplied by the coordinate differentials, I think. The Lagrangian density incorporates matter-energy fields. ψ can be ?, or F, etc. For special cases, L is given by:



For the fluid, ε0 is the rest frame energy density (in principle known from the equation of state, and FWIW, dust would have ε0 = mn0c2), n0 the rest frame number density, s0 the rest frame entropy density (s0 = 0 for dust), and υμ is the four velocity. The last two terms in the Lagrangian density are to enforce the continuity equation for the fluid, as well as the four-velocity normalization to -c2. We’d maximize S w/r to gμν, n0, s0, and υμ. Combining the resultant equations we should get Einstein’s equation, as well as the 4-momentum (more or less, energy and momentum) and entropy fluid flow equations. This checks out, unknown-wise. In the fluid case, we have 6 unknowns: gμν, n0, s0, υμ, λ(x), ζ(x). And we have 6 equations from minimization w/r to gμν, n0, s0, and υμ, and the two constraints [and of course we presume to know ε0(n0, s0) in the first place]. Likewise in the EM case (no charges in this expression, just field), we have two unknowns: gμν, Fμν (really Aμ where Fμν = ∂μAν - ∂νAμ) and two equations from minimization w/r to gμν and Aμ. Minimization of S w/r to gμν gives us Einstein’s equation and minimization w/r to Aμ gives us ∙ = 0 I believe.

So to solve Einstein’s equation, we’d first postulate some generic coordinate system, a book-keeping coordinate system. Often our problems will have radial symmetry and so we could use t, r, θ, φ. A physical picture we can put to this is that of a coordinate system with a clock ticking at every coordinate. We might imagine t to be the time at ∞, say, far from any mass-energy. We could actually calibrate clocks to this time if we wanted to, as we’ll discuss later. But for now, suffice to say it’s one possible example for t. r would be the radial coordinate obviously, but could be something like – count how many steps I make to get to certain point and that is the radial coordinate. We don’t need anything more accurate than that. θ and φ could be done with a protractor, say.

Note that we may use symmetry considerations to postulate likely forms for gμν, ε0(n0, s0), and υ. For instance, for a static spherically symmetric star, we might make the following ansatz:



and if it’s rotating axially at a constant speed, we might (do) make the ansatz:

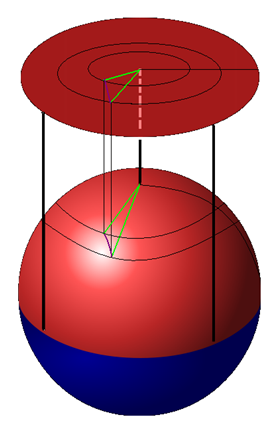


Then we’d fill these guys in and solve for the unknown functions (I imagine for the rotating star situation, we’d use the boundary condition that the physical rotation at the surface, measured in time infinitely far away is known).

Should note that for a given matter-energy distribution we can have many different equivalent solutions to Einstein’s equations (i.e. can express the metric in many different coordinate systems). For instance, we can do a kind of Cartesian coordinate system, or a spherical coordinate system, etc. Will also throw in that if we solve for gμν in one coordinate system, doing a simple coordinate transformation to another coordinate system will not give us gμν appropriate for that system, necessarily, as Einstein’s equations are non-linear and don’t map to themselves upon a coordinate transformation. But for free space do a coordiante transformation like this would work because Einstein’s equations reduce to Rμν = 0.

**Interpretation of the Metric**

Okay so this metric then tells us how to construct the space-time interval for any differential coordinate change. For example, in the image below, we could have polar coordinates mapped onto a hemisphere. The metric tells us how the space-time interval relates to the coordinate change.

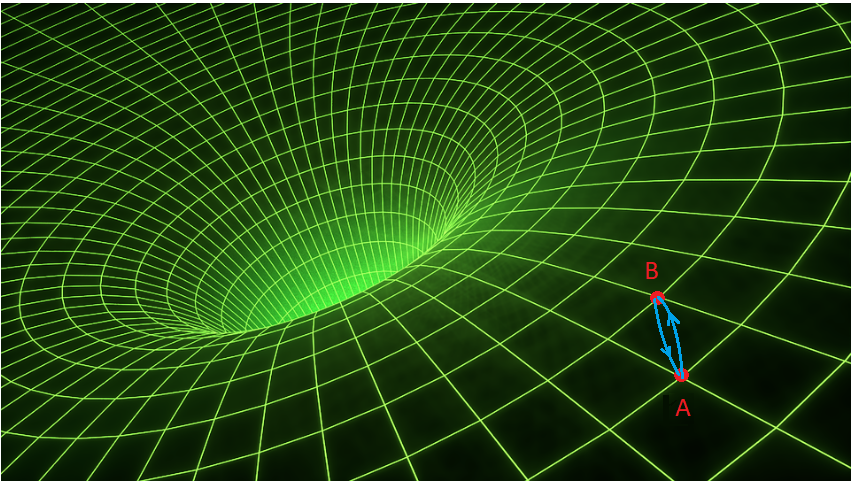


But then we have a further question. We’re used to breaking space-time intervals into distance and time. So can we do that here too? How much time has elapsed during a space-time interval ds, and how much space? We didn’t have to worry about this per se´ in special relativity as the metric and Lorentz transformation equations were already written in terms of the physical coordiantes. Nonetheless, if we hadn’t presumed so, we could’ve applied back then the same techniques we’ll apply now, to identify them as such. So consider a general metric,



where g is a symmetric matrix (it must be), and dx0 = cdt of course. The coefficients gαβ can be interpreted as α·β, where these are the covariant basis vectors in those respective directions.

These xα are called book-keeping coordinates, to distinguish them from any actual physical coordinate. These are just coordinates on a manifold, like the radial coordinates were on the 2-sphere in the tensor notes. And their intervals dx0, or dxidxi do not necessarily equate to any physical time or space interval. Still, at any given point in our manifold, according to an observer fixed at a given book-keeping coordinate, a book-keeping coordinate displacement is commensurate to some physical time and space displacement, which we’ll call d and d. And we’d like to know what it is.



First, a preliminary result. Let’s see how proper time is related to book-keeping time.

This will basically tell us how fast the clocks are ticking at each point in space. So consider a spatial point in our manifold, and an event that begins and ends at that same spatial point, like a clock at a point in space ticking away the seconds. Our metric would describe the invariant interval associated with this event as:



Now remember that our space is everywhere locally flat, and so can be locally described by a Minkowski metric. So transform to such a set of coordinates. In this reference frame, locally at rest, the space-time distance associated with this space time event (say two ticks of our clock) would be described as:



Equating the two, we have an equation that relates proper local time to book-keeping time,



So the book keeping time clocks and proper time clocks at a given coordinate are directly proportional. And we could easily change to a coordinate system where clocks at each spatial coordinate tick off proper time seconds instead of book-keeping time seconds. I think we can synchronize all clocks at t = 0 (at least for metrics w/o an event horizon?) say, but then thereafter they will diverge if the metric isn’t flat. One last thing, note that g00 typically goes to -1 as we move far away from mass or energy, and so we could interpret book-keeping time as the proper time for clocks far from any mass-energy. Can also note that as |g00| gets smaller (as it typically does closer to matter-energy – see Star), a given interval of local proper time will comprise more and more intervals of book-keeping time. So for instance, if we had clocks that were keeping track of book-keeping time at every point in space, they would seem to run faster and faster the closer we get to a star (i.e., the smaller |g00| gets). Is it possible to have a clock telling book-keeping time everywhere? Well, we could station a clock at every coordinate. We could calibrate them so that they read out dt = dτ/√(-g00) instead of dτ (dτ is what they’d naturally do). Then to synchronize them at t = 0, we could take a light pulse at some reference point far away. That would signal t = 0. And as soon as the light pulse hits the clocks they would start. Then, all we need is to do is add the book-keeping time Δtthat it takes for the light-wave to get from our reference point to the clock at the given coordinate. Well this can be obtained from ds2 = 0 (true for any light wave). So we’d just solve for dt in terms of dxj and integrate over the path that light would take from the reference point to the clock to get Δt. This might only work for non-time-dependent metrics.

But now let’s consider physical distance. We’ll look for the distance between two space-time points differentially far apart. We’ll call them A and B (see diagram above). We know, by going back to our local Minkowski metric, that a light ray would be locally described as traveling at the speed of light, and so the distance between AB can be ascertained as d = cd where d is the physical time it takes for a light ray to go from A to B. This would just be the physical time described by the interval dx0. Alas, we don’t know how physical time relates to book keeping time yet. But there’s another way; we can formulate the distance in terms of the proper time quantity we already know how to calculate. Let A and B be at xj and xj + dxj respectively (j stands for spatial coordinates). If we send a light beam from A to B, and then have it immediately sent back/reflected to A, we can say the physical distance between A and B is d = cdτ/2 = c√(-g00)dt/2, where dτ is the time it takes for the signal to emanate from and return to point A, according to the local frame, and dt is the corresponding change in book-keeping coordinates, as we saw. But how do we relate dt to dxj? We get this from the fact that in our local Minkowski frame, the light beam path will satisfy the equation ds2 = cdt2 – d**x**∙d**x** = 0 (by virtue of the fact that it travels at the speed of light), and so this will also be true when transforming back to our book-keeping coordinates. Going back to our original frame then, we’d have, for light’s outward journey,



and for the backward journey, setting d**x**BA = -d**x**AB, we have:



adding the two times together we get:



So the distance between these two points is:



Dropping the subscripts, we can say that the physical space metric is given by:



FWIW, far from any matter-energy, d2 will just be, at least in Cartesian coordinates, dxjdxj. Just like we were able to place at all the coordinates a clock which told book-keeping time, we could label all the coordinates with their coordinate distance. We could just start at the origin of our coordinate system, say, make a displacement dxi, measure d2 with a physical ruler and calculate the value of dxi from the highlighted formula above. Or we could start at some point away from the origin, label that distance, ‘15’, say, and then procede as before. Or we could start somewhere far away, call that ‘15’, and procede backwards – towards the origin, say.

Now let’s consider physical time. Physical time between events at the same spatial coordinates is just proper time, which we already found. But now we’ll examine how to measure time between events that aren’t at the same spatial point. We’ll consider two points/events A and B again. But this time, their times won’t be connected as they were before (they were on the so-called light cone such that ds2 = 0), because we don’t want to restrict ourselves to events that can be connected only by light. So book-keeping times will be arbitrary, though differentially small. So we have between A and B just some generic (dxAB0, dxABj). We can get the physical time difference between these events by comparing the ds2 between A and B (leaving off the subscripts) as expressed in book-keeping coordinates (LHS) and physical coordinates (RHS):



and so we have:



and finally,



Can see that when dxi = 0, then d = dτ. And I’ve been thinking of dτ as the physical time ticking off at each coordinate. But really d should be that, right, as it’s the physical time for the reference frame? So why does dxi make a difference? We’ll note that it only makes a difference if gi0 is non-zero. These elements would couple time to space, like we have, for instance, in special relativity with a moving reference frame. And I think the issue here then, is simultaneity. If we go back to SR, we’ll see that the dxi term is exactly the the delta t term that shows up in the Lorentz transformation for a reference frame moving with velocity v. It’s also the term that shows up in the equation for the time separation between two events, simultaneous in one frame, as viewed from a reference frame traveling with velocity v w/r to that frame in which they’re simultaneous. So I guess we could think then of the clocks indeed ticking off physical time according to dτ, but offset in a sense by an amount proportional to dxi, that comes from the relativity of simultaneity.

Okay, now as explicitly invoked above, we can now write the metric as:



For what it’s worth, at any book-keeping coordinate, one can now define ‘physical’ coordinates which diagonalize the metric, enabling us to write it in Minkowski form:



where specifically,



(basically saying that the physical spatial displacements are root eigenvalue times corresponding eigenvector) Then dxj = (1/√Gk)Ujkdk. And,



Then in terms of these new physical coordinates, our metric will be:



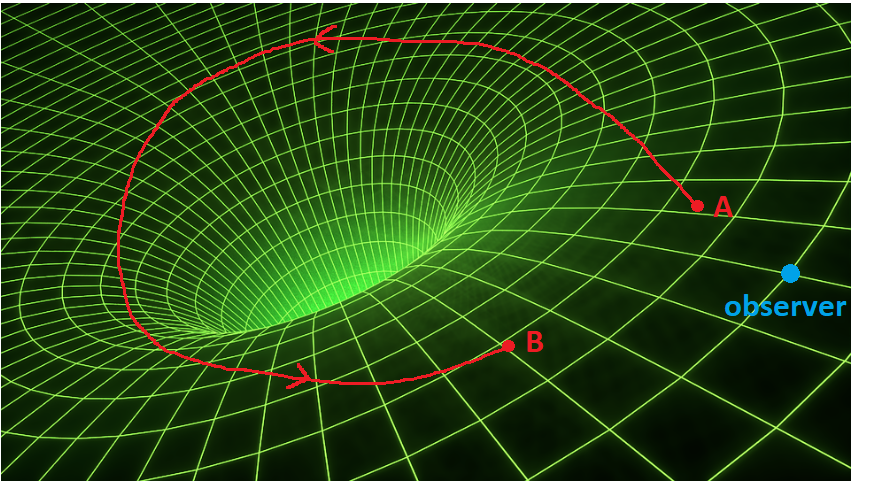
as written more compactly just above. So I think we can reformulate the process of finding physical coordinates as just the process of going to a local frame of reference that diagonalizes (Minkowski-izes) the metric. And FWIW, our formulas above tell us how to find these coordinates - well, a special set of such coordinates. This appears to be an at rest, inertial (i.e., instantaneously free falling) reference frame. But we can find others that diagonalize the metric, such as any local Lorentz transformation to a local moving frame (our manipulations above have implicitly presumed a frame that is *static* w/r to the given manifold/coordinate system). So the differential physical time and physical distance are what a locally at rest freely falling person would measure for the given space-time differential displacement. If we want the physical time and distance for an observer *moving at a non-zero velocity w/r to the coordinate frame*, then I’d imagine we could just use the usual special relativity Lorentz transformation stuff, or more directly, the d´ = d/γ (b/c moving person would measure proper time, right?) d´ = d/γ (because stationary person would measure proper distance, right?) formulas. We could use this to get the elapsed time, for instance, of the person who is doing the traveling, and I guess the perceived distance as well.

The physical coordinate transformation highlighted above *is* just a local transformation though. However, if it turns out that d0 is an *exact* differential, then that means the change in physical time between two events is only dependent on the space-time coordinates of the two events, not on the space-time path that was taken to get there. So for instance if a star’s d0 were exact, then that would mean the elapsed time for a person to travel around the star (could travel in radial direction and stuff too) and back to their starting point would be the same as for a person who never left. Also, if we have an exact differential, then we can write the physical time as a function of book-keeping time and space, = (*t*,**x**). In that case there are hypersurfaces in 4-space at the same physical time, hypersurfaces of simultaneity and we can define this as global time. We can can make a coordinate transformation, *t* = *t*(,**x**), to this reference frame, whereby clocks at every point progress at the same rate – so there is no book-keeping time dilation. Furthermore, in terms of the variables 0, **x**i we can then write the metric as:



One can apparently do this for a special few metrics, like the Robertson-Walker metric which defines the structure of space-time on a universal scale. But we cannot do this for cases like a Schwarzild metric, which describes the space in and around a star.

A couple questions now… Consider some path through space xα(s) that begins and ends at xAα and xBα respectively.



What is the physical length of this path? This would seem to be:



where gαβ(s) is parameterized by s through g’s dependence on the book-keeping coordinates xα(s). Now how long would this path take to be traversed, according to the path-taker? This would be:



Another question, ‘how long would this appear to take to be traversed to an observer elsewhere’? In this case, I think we would use the *observer’s* path through space-time xα(s), in place of the object’s path, and integrate *their* physical time between the same two limits.



Typically, the observer wouldn’t be going anywhere, and so this would simplify to:



And if it were the case that g00 is time-independent, like with the Schwarzild metric, then this would just be:



g00 *would* typically depend on the (observer’s) position.